ON WEIGHTED REMAINDER FORM OF HARDY-TYPE INEQUALITIES

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Abstract. We use different approaches to study a generalization of a result of Levin and Stečkin concerning an inequality analogous to Hardy's inequality. Our results lead naturally to the study of weighted remainder form of Hardy-type inequalities.

1. Introduction

Let $p>1$ and l^p be the Banach space of all complex sequences $\mathbf{a}=(a_n)_{n\geq 1}$. The celebrated Hardy's inequality [13, Theorem 326] asserts that for $p>1$ and any $\mathbf{a}\in l^p$,

$$\sum_{n=1}^{\infty}\Big|\frac{1}{n}\sum_{k=1}^{n}a_k\Big|^p \leq \Big(\frac{p}{p-1}\Big)^p\sum_{k=1}^{\infty}|a_k|^p. \tag{1.1}$$

Hardy's inequality can be regarded as a special case of the following inequality:

$$\Big|\Big|C\cdot\mathbf{a}\Big|\Big|_p^p=\sum_{n=1}^{\infty}\Big|\sum_{k=1}^{\infty}c_{n,k}a_k\Big|^p\leq U_p\sum_{n=1}^{\infty}|a_n|^p,$$

in which $C=(c_{n,k})$ and the parameter $p>1$ are assumed fixed, and the estimate is to hold for all complex sequences $\mathbf{a}\in l^p$. The l^p operator norm of C is then defined as

$$||C||_{p,p}=\sup_{||\mathbf{a}||_p\leq 1}\Big|\Big|C\cdot\mathbf{a}\Big|\Big|_p.$$

Hardy's inequality thus asserts that the Cesáro matrix operator $C=(c_{j,k})$, given by $c_{j,k}=1/j, k\leq j$ and 0 otherwise, is bounded on l^p and has norm $\leq p/(p-1)$. (The norm is in fact $p/(p-1)$.) Hardy's inequality leads naturally to the study on l^p norms of general matrices. For example, we say a matrix $A=(a_{j,k})$ is a weighted mean matrix if its entries satisfy $a_{j,k}=0, k>j$ and

$$a_{j,k}=\lambda_k/\Lambda_j,\ 1\leq k\leq j;\Lambda_j=\sum_{i=1}^{j}\lambda_i,\lambda_i\geq 0,\lambda_1>0.$$

There are many studies on the l^p operator norm of a weighted mean matrix and we refer the reader to the articles [1]-[3], [6]-[11] and the references therein for more results in this area.

In this paper, we are interested in the following analogue of Hardy's inequality, given as Theorem 345 of [13], which asserts that the following inequality holds for $0<p<1$ and $a_n\geq 0$ with $c_p=p^p$:

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=n}^{\infty}a_k\Big)^p\geq c_p\sum_{n=1}^{\infty}a_n^p. \tag{1.2}$$

It is noted in [13] that the constant $c_p=p^p$ may not be best possible and a better constant was indeed obtained by Levin and Stečkin [15, Theorem 61]. Their result is more general as they proved, among other things, the following inequality ([15, Theorem 62]), valid for $0<p\leq 1/3, r\leq p$ or $1/3<p<1, r\leq 1-2p$ (note that this is given in [15] as $1/3<p<1, r\leq (1-p)^2/(1+p)$ but

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an inspection of the proof of Theorem 62 in [15] shows that they actually proved their result for $1/3 < p < 1, r \leq 1-2p$, see especially the proof of Lemma 3 in the proof of Theorem 62 in [15] for this) with $a_n \geq 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n^r}\Big(\sum_{k=n}^{\infty} a_k\Big)^p \geq c_{p,r} \sum_{n=1}^{\infty} \frac{a_n^p}{n^{r-p}}, \tag{1.3}$$

where the constant $c_{p,r} = (p/(1-r))^p$ is best possible (see for example, [12]). It follows that inequality (1.2) holds for $0 < p \leq 1/3$ with the best possible constant $c_p = (p/(1-p))^p$.

The above result of Levin and Stečkin has been studied in [8] and [12]. In [8], a simple proof of inequality (1.3) for the case $0 < r = p \leq 1/3$ is given. In [12], inequality (1.3) is shown to hold for $0 < r = p \leq 0.346$.

It is our goal in this paper to first generalize the above result of Levin and Stečkin. We make a convention in this paper that for any integer $k \geq 1$, $((k+1)^0 - k^0)/0 = \ln((k+1)/k)$ and we shall prove in Section 3 the following

Theorem 1.1. *Let $a_n > 0, 0 < p < 1$. The following inequality holds for any number r satisfying $(2+r)p \leq 1$,*

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n^r}\sum_{k=n}^{\infty}\Big(\frac{(k+1)^r-k^r}{r}\Big)a_k\Big)^p \geq \Big(\frac{p}{1-rp}\Big)^p \sum_{n=1}^{\infty} a_n^p. \tag{1.4}$$

The above inequality reverses when $p \geq 1, 1/p-2 \leq r < 1/p$ or $p < 0$, $1/p - 2 \leq r < 1/p$. The constant is best possible.

One can show following the construction in [12] that the constant in (1.4) is best possible. We let q be the number defined by $1/p + 1/q = 1$ and note that by the duality principle (see [16]), the statement of Theorem 1.1 is equivalent to the following

Theorem 1.2. *Let $a_n > 0, 0 < p < 1$. The following inequality holds for any number r satisfying $(2+r)p \leq 1$,*

$$\sum_{n=1}^{\infty}\Big(\Big(\frac{(n+1)^r-n^r}{r}\Big)\sum_{k=1}^{n}\frac{a_k}{k^r}\Big)^q \leq \Big(\frac{p}{1-rp}\Big)^q \sum_{n=1}^{\infty} a_n^q. \tag{1.5}$$

The above inequality also holds when $p > 1, 1/p-2 \leq r < 1/p$ and the reversed inequality (1.5) holds when $p < 0$, $1/p-2 \leq r < 1/p$. The constant is best possible.

We now write $r = \alpha + \beta/p$ in Theorem 1.1 and note that for $\beta \leq 0$, we have

$$n^{\beta/p}\Big(\frac{(n+1)^{\alpha}-n^{\alpha}}{\alpha}\Big) \geq \Big(\frac{(n+1)^{\alpha+\beta/p}-n^{\alpha+\beta/p}}{\alpha+\beta/p}\Big).$$

This combined with inequality (1.4) allows us to deduce the following (via a change of variables $a_n \to n^{-\beta/p}a_n$)

Corollary 1.1. *Let $a_n \geq 0, \beta \leq 0 < \alpha, 0 < p < 1$. The following inequality holds for $0 < p \leq (1-\beta)/(2+\alpha)$,*

$$\sum_{n=1}^{\infty}\frac{1}{n^{\beta}}\Big(\frac{1}{n^{\alpha}}\sum_{k=n}^{\infty}\Big((k+1)^{\alpha}-k^{\alpha}\Big)a_k\Big)^p \geq \Big(\frac{\alpha p}{1-\beta-\alpha p}\Big)^p \sum_{n=1}^{\infty}\frac{a_n^p}{n^{\beta}}.$$

The constant is best possible.

One can also deduce the cases $0 < p \leq 1/3, r \leq p$ or $1/3 < p < 1, r \leq 1-2p$ of inequality (1.2) via similar transformations of inequality (1.4).

The case $r = 0$ in Theorem 1.1 implies the following

Corollary 1.2. *Let $a_n \geq 0$. For $0 < p \leq 1/2$, we have*

$$\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\ln\Big(\frac{k+1}{k}\Big)a_k\Big)^p \geq p^p\sum_{n=1}^{\infty}a_n^p.$$

The constant is best possible.

Note that as $\ln((k+1)/k) \leq 1/k$, Corollary 1.2 implies the following well-known Copson's inequality [13, Theorem 344] when $0 < p \leq 1/2$:

$$\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{a_k}{k}\Big)^p \geq p^p\sum_{n=1}^{\infty}a_n^p.$$

Similarly, the case $r = 0$ in Theorem 1.2 implies the following

Corollary 1.3. *Let $a_n > 0$. For $-1 \leq p < 0$, we have*

$$\sum_{n=1}^{\infty}\Big(\ln\Big(\frac{n+1}{n}\Big)\sum_{k=1}^{n}a_k\Big)^p \leq \Big(\frac{p}{p-1}\Big)^p\sum_{n=1}^{\infty}a_n^p.$$

The constant is best possible.

We point out here that Corollary 1.3 implies the well-known Knopp's inequality [14, Satz IV] (which is inequality (1.1) with $p < 0$ and $a_n > 0$) when $-1 \leq p < 0$.

We note that it is pointed out in [12] that inequality (1.2) can not hold for all $0 < p < 1$ with the constant c_p being $(p/(1-p))^p$. However, Levin and Stečkin [15, Theorem 61] was able to improve the constant $c_p = p^p$ for all $0 < p < 1$ as their result is given in the following:

Theorem 1.3. *Inequality* (1.2) *holds with c_p being*

$$c_p = \begin{cases} \Big(\frac{p}{1-p}\Big)^p, & 0 < p \leq 1/3; \\ \frac{1}{2}\Big(\frac{1+p}{1-p}\Big)^{1-p}, & 1/3 < p \leq 3/5; \\ 2\Big(\frac{p}{3-p}\Big)^p, & 3/5 \leq p < 1. \end{cases}$$

Our method in this paper allows us to give another proof of the above result. In fact, we shall prove the following result in Section 4:

Theorem 1.4. *Let $0 < p < 1, 0 < r \leq p$. Inequality* (1.3) *holds with $c_{p,r}$ with*

$$c_{p,r}^{-1} = (2-p+r)\Big(\frac{1-p}{1-p+2r}\Big)^{1-p}.$$

We note here the constant $c_{p,r}^{-1}$ in the statement of Theorem 1.4 is nothing but the constant $\chi(r)$ defined in Lemma 5 in the proof of Theorem 62 in [15]. We now say a few words on how to deduce Theorem 1.3 from Theorem 1.4, this is also given in Theorem 62 of [15]. First, it is easy to show that for fixed p, $c_{p,r}^{-1}$ is minimized at $r = (3-2p)(1-p)/2p$. When $1/3 \leq p \leq 3/5$, we have $p \leq (3-2p)(1-p)/2p$, hence on setting $r = p$ in Theorem 1.4 implies the corresponding cases of Theorem 1.3. When $3/5 \leq p < 1$, we have $p \geq (3-2p)(1-p)/2p$ and a combination of Lemma 7 in the proof of Theorem 62 in [15] and setting $r = (3-2p)(1-p)/2p$ in Theorem 1.4 implies the corresponding cases of Theorem 1.3.

Our method in proving Theorem 1.3 and Theorem 1.4 is more flexible and there is still room to further improve the constant c_p or $c_{p,r}$, when they are not best possible. In this paper, we shall only consider the constant $c_{1/2}$ of the special case $p = 1/2$ in (1.2). It's given as $1/\sqrt{2}$ in [13] and was improved to be $\sqrt{3}/2$ by Levin and Stečkin in [15]. The author has shown in [8] that one can take $c_{1/2} = 0.8967$ but at that time he was not aware that Boas and de Bruijn [4] showed that

$15/17 \approx 0.8824 < c_{1/2} < 1/1.08 \approx 0.9259$ and De Bruijn [5] showed that $c_{1/2} \approx 1/1.1064957714 \approx 0.90375$. With less effort than de Bruijn's analysis in [5], we shall use our approach in this paper to show in Section 4 that we can take $c_{1/2}$ to be 0.9, which coincides with the optimal $c_{1/2}$ for the first two decimal expansions. This is given in the following

Theorem 1.5. *Inequality* (1.4) *holds when* $p = 1/2$ *with* $c_{1/2} = 0.9$.

We note the following result:

Theorem 1.6. *Let* $a_n > 0$ *and* $\alpha > 0$. *Then for* $p < 0$ *or* $p \geq 1$ *and* $\alpha p > 1$, *we have*

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n^{\alpha}}\sum_{k=1}^{n}(k^{\alpha}-(k-1)^{\alpha})a_k\Big)^p \leq \Big(\frac{\alpha p}{\alpha p-1}\Big)^p\sum_{k=1}^{\infty}a_k^p. \tag{1.6}$$

The constant is best possible.

The special case $p > 1, \alpha \geq 1, \alpha p > 1$ of inequality (1.6) was proved by the author in [6]. The general cases of inequality (1.6) were proved by Bennett in [3].

It's easy to show that we have, for $\alpha > 0, r \geq 1$, $p \geq 1$ that when $k \geq 1$,

$$k^{(1-r)/p}\Big(\frac{k^{\alpha}-(k-1)^{\alpha}}{\alpha}\Big) \leq \frac{k^{\alpha+(1-r)/p}-(k-1)^{\alpha+(1-r)/p}}{\alpha+(1-r)/p}.$$

The above inequality reverses when $p < 0$. Replacing α with $\alpha+(1-r)/p$ in (1.6) and applying the above inequality, we deduce immediately the following (via a change of variables $a_n \rightarrow n^{(r-1)/p}a_n$) result ([3, Theorem 1]):

Corollary 1.4. *Let* $a_n > 0$. *Suppose that* $\alpha > 0$ *and* $r \geq 1$. *Then for* $p < 0$ *or* $p \geq 1$ *and* $\alpha p > r$, *we have*

$$\sum_{n=1}^{\infty}n^{r-1}\Big(\frac{1}{n^{\alpha}}\sum_{k=1}^{n}(k^{\alpha}-(k-1)^{\alpha})a_k\Big)^p \leq \Big(\frac{\alpha p}{\alpha p-r}\Big)^p\sum_{k=1}^{\infty}k^{r-1}a_k^p.$$

The constant is best possible.

We point out here that we will present two proofs of Theorem 1.1 in Section 3. The first one can be viewed as an analogue to Bennett's proof of Theorem 1.6 and the second one is a generalization of the proof of inequality (1.3) given in [15]. One then asks whether one can adapt the approach used in the second proof of Theorem 1.1 to give another proof of Theorem 1.6 and this is indeed possible as we will give an alternative proof of Theorem 1.6 in Section 5.

As it is pointed out in [3] that inequality (1.6) fails to hold when $\alpha p \leq 1$. One therefore wonders whether there are any analogues of inequality (1.6) that hold when $\alpha p \leq 1$. For this we note that it follows from Theorem 1.1 that the reversed inequality (1.4) holds when $p \geq 1$ or $p < 0$ under certain restrictions on r. One may view these reversed inequalities as the $\alpha p \leq 1$ analogues to inequality (1.6). However, the duality principle also allows one to interpret these inequalities as $\alpha p > 1$ (with a different α) analogues to (1.6). To see this, we take the $p > 1$ case in Theorem 1.1 as an example and we use its dual version, Theorem 1.2 with $p > 1, 1/p - 2 \leq r < 1/p$. We interchange the variables p and q and replace r by $-r$ to recast inequality (1.5) for this case as ($1/p - 1 < r \leq 1 + 1/p$):

$$\sum_{n=1}^{\infty}\Big(\Big(\frac{n^{-r}-(n+1)^{-r}}{r}\Big)\sum_{k=1}^{n}k^r a_k\Big)^p \leq \Big(\frac{p}{(r+1)p-1}\Big)^p\sum_{n=1}^{\infty}a_n^p.$$

Note that the above inequality is analogue to inequality (1.6) in the sense that we have $(r+1)p > 1$ here. We can further recast the above inequality as

$$\sum_{n=1}^{\infty}\Big(\Big(\frac{n^{-r}-(n+1)^{-r}}{r}\Big)(1+r)\Big(\sum_{i=1}^{n}i^r\Big)\frac{1}{\sum_{i=1}^{n}i^r}\sum_{k=1}^{n}k^r a_k\Big)^p \leq \Big(\frac{(r+1)p}{(r+1)p-1}\Big)^p\sum_{n=1}^{\infty}a_n^p. \tag{1.7}$$

The above inequality and inequality (2.3) below imply immediately the following inequality for $a_n > 0, p > 1, 2 \le \alpha \le 2 + 1/p$:

$$\sum_{n=1}^{\infty}\Big(\frac{1}{\sum_{i=1}^{n} i^{\alpha-1}}\sum_{i=1}^{n} i^{\alpha-1}a_i\Big)^p \le \Big(\frac{\alpha p}{\alpha p-1}\Big)^p\sum_{n=1}^{\infty}a_n^p. \tag{1.8}$$

The above inequality has been studied in [6], [3], [8], [10] and [11]. The author [6] and Bennett [3] proved inequality (1.8) for $p > 1, \alpha \ge 2$ or $0 < \alpha \le 1, \alpha p > 1$ independently. The author [8] has shown that (1.8) holds for $p \ge 2, 1 \le \alpha \le 1 + 1/p$ or $1 < p \le 4/3, 1 + 1/p \le \alpha \le 2$. Recently, the author [11] has shown that inequality (1.8) holds for $p \ge 2$, $0 \le \alpha \le 1$. In [10, Corollary 2.4], it is shown that inequality (1.8) holds for $\alpha > 0, p < 0$.

Other than the above point of view of the reversed inequality of (1.4) using the duality principle, we may also regard the (reversed) inequality of (1.4) as a type of "weighted remainder form of Hardy-type inequalities", a terminology we use after Pečarić and Stolarsky, who studied a special case of this type of inequalities in [17, Sec 3]. Theorem 1.1 thus leads naturally to the study of the following weighted remainder form of Hardy-type inequalities in general:

$$\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{\lambda_k a_k}{\Lambda_n}\Big)^p \le \Big(\frac{p}{p-L}\Big)^p\sum_{n=1}^{\infty}a_n^p, \tag{1.9}$$

where (λ_n) is a positive sequence satisfying $\Lambda_n = \sum_{k=n}^{\infty}\lambda_k < +\infty$ and L is a number such that $L < p$ when $p > 0$ and $L > p$ when $p < 0$. We want the above inequality to hold for $p > 1$ or $p < 0$ and any positive sequence (a_n) satisfying $\sum_{n=1}^{\infty}a_n^p < +\infty$. We also want the reversed inequality of (1.9) to hold when $0 < p < 1$. We shall study inequality (1.9) in Section 6. We shall find conditions on the λ_n's so that inequality (1.9) (or its reverse) can hold under these conditions.

2. A heuristic approach to inequality (1.4)

In this section, we first give a heuristic approach towards establishing inequality (1.4). This approach will provide motivation and serve as a guideline for our proof of Theorem 1.1 later. In fact, the approach we discuss here is in some sense a "natural" approach towards establishing Hardy-type inequalities. For simplicity, we consider inequality (1.4) for $0 < p < 1, r > 0, rp < 1$. A general approach towards establishing the above inequality is to apply the reversed Hölder's inequality to get

$$\Big(\sum_{k=n}^{\infty}\Big((k+1)^r-k^r\Big)a_k\Big)^p \ge \Big(\sum_{k=n}^{\infty}W_k^{1/(1-p)}\Big)^{p-1}\Big(\sum_{k=n}^{\infty}W_k\big((k+1)^r-k^r\big)^p a_k^p\Big), \tag{2.1}$$

where (W_k) is a sequence to be determined. A general discussion on Hardy-type inequalities in [11] implies that one can in fact obtain the best possible constant on choosing W_k properly.

We now give a description of one choice for the W_k's in (2.1). In fact, more naturally, we write $\big((k+1)^r - k^r\big)a_k = \big((k+1)^r - k^r\big)k^{-\gamma}k^{\gamma}a_k$, so that by the reversed Hölder's inequality (and one can reconstruct the W_k's from this), we have

$$\Big(\sum_{k=n}^{\infty}((k+1)^r-k^r)a_k\Big)^p \ge \Big(\sum_{k=n}^{\infty}\big((k+1)^r-k^r\big)^{p/(p-1)}k^{-\gamma p/(p-1)}\Big)^{p-1}\Big(\sum_{k=n}^{\infty}k^{\gamma p}a_k^p\Big),$$

where $\gamma < r-1/p < 0$ (this guarantees the finiteness of the two factors of the right-hand side expressions above) is a parameter to be chosen later. Using this, we then have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big(\frac{1}{n^r}\sum_{k=n}^{\infty}\Big(\frac{(k+1)^r-k^r}{r}\Big)a_k\Big)^p \\
\geq\ &\sum_{n=1}^{\infty}\frac{\Big(\sum_{k=n}^{\infty}\big((k+1)^r-k^r\big)^{p/(p-1)}k^{-\gamma p/(p-1)}\Big)^{p-1}\Big(\sum_{k=n}^{\infty}k^{\gamma p}a_k^p\Big)}{r^pn^{rp}} \\
=\ &\sum_{k=1}^{\infty}a_k^pk^{\gamma p}\sum_{n=1}^{k}\frac{\Big(\sum_{i=n}^{\infty}\big((i+1)^r-i^r\big)^{p/(p-1)}i^{-\gamma p/(p-1)}\Big)^{p-1}}{r^pn^{rp}}.
\end{aligned}$$

Asymptotically, we have

$$\begin{aligned}
&\sum_{i=n}^{\infty}\big((i+1)^r-i^r\big)^{p/(p-1)}i^{-\gamma p/(p-1)} \\
\sim\ &r^{p/(p-1)}\sum_{i=n}^{\infty}i^{(r-1-\gamma)p/(p-1)} \\
\sim\ &\frac{r^{p/(p-1)}n^{(r-1-\gamma)p/(p-1)+1}}{(\gamma+1-r)p/(p-1)-1}.
\end{aligned} \tag{2.2}$$

It follows that asymptotically, we have

$$\begin{aligned}
&k^{\gamma p}\sum_{n=1}^{k}\frac{\Big(\sum_{i=n}^{\infty}\big((i+1)^r-i^r\big)^{p/(p-1)}i^{-\gamma p/(p-1)}\Big)^{p-1}}{r^pn^{rp}} \\
\sim\ &\frac{1}{((\gamma+1-r)p/(p-1)-1)^{p-1}}k^{\gamma p}\sum_{n=1}^{k}\frac{1}{n^{1+\gamma p}} \\
\sim\ &-\frac{1}{((\gamma+1-r)p/(p-1)-1)^{p-1}}\frac{1}{\gamma p}.
\end{aligned}$$

We then want to choose γ so that the last expression above is maximized and calculation shows that in this case we need to take $\gamma=(rp-1)/p^2(<r-1/p)$ and the so taken γ makes the value of the last expression above being exactly the constant appearing on the right-hand side of (1.4).

The above approach can be applied to discuss inequality (1.5) similarly and in this case, we can make our argument rigorous to give a proof of Theorem 1.2.

Proof of Theorem 1.2:

Due to the similarities of the proofs (taken into account the reversed inequality of (2.3)), we may assume $0<p<1$ here. By the reversed Hölder's inequality, we have

$$\Big(\sum_{k=1}^{n}\frac{a_k}{k^r}\Big)^q\leq\Big(\sum_{k=1}^{n}\frac{k^{\gamma}a_k^q}{k^{rq}}\Big)\Big(\sum_{k=1}^{n}k^{\gamma/(1-q)}\Big)^{q-1},\quad \gamma=1/p+r/(p-1).$$

It follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big(\Big(\frac{(n+1)^r-n^r}{r}\Big)\sum_{k=1}^{n}\frac{a_k}{k^r}\Big)^q \\
\leq\ &\sum_{k=1}^{\infty}\frac{k^{\gamma}a_k^q}{k^{rq}}\sum_{n=k}^{\infty}\Big(\frac{(n+1)^r-n^r}{r}\Big)^q\Big(\sum_{i=1}^{n}i^{\gamma/(1-q)}\Big)^{q-1}.
\end{aligned}$$

We now note the following inequality ([15, Lemma 2, p. 18]), which asserts for $r \geq 1$, we have

$$\sum_{i=1}^{n} i^r \geq \frac{r}{1+r}\frac{n^r(n+1)^r}{(n+1)^r-n^r}. \tag{2.3}$$

The above inequality reverses when $-1 < r \leq 1$ (only the case $r \geq 0$ of the above inequality was proved in [15] but one checks easily that the proof extends to the case $r > -1$).

When $\gamma/(1-q) \geq 1$, which is equivalent to the condition $(r+2)p \leq 0$, we can apply estimation (2.3) to get

$$\sum_{i=1}^{n} i^{\gamma/(1-q)} \geq \frac{1}{1+\gamma/(1-q)}\Big(\int_n^{n+1} x^{-\gamma/(1-q)-1}dx\Big)^{-1} \tag{2.4}$$

This combines with (2.4) implies that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big(\Big(\frac{(n+1)^r-n^r}{r}\Big)\sum_{k=1}^{n}\frac{a_k}{k^r}\Big)^q \\
\leq\ & \Big(\frac{1}{1+\gamma/(1-q)}\Big)^{q-1}\sum_{k=1}^{\infty}\frac{k^\gamma a_k^q}{k^{rq}}\sum_{n=k}^{\infty}\Big(\int_n^{n+1} x^{r-1}dx\Big)^q\Big(\int_n^{n+1} x^{-\gamma/(1-q)-1}dx\Big)^{1-q} \\
\leq\ & \Big(\frac{1}{1+\gamma/(1-q)}\Big)^{q-1}\sum_{k=1}^{\infty}\frac{k^\gamma a_k^q}{k^{rq}}\sum_{n=k}^{\infty}\int_n^{n+1} x^{q(r-1)+(1-q)(-\gamma/(1-q)-1)}dx \\
=\ & \Big(\frac{1}{1+\gamma/(1-q)}\Big)^{q} = \Big(\frac{p}{1-rp}\Big)^q.
\end{aligned}$$

This completes the proof of Theorem 1.2.

3. Proof of Theorem 1.1

We shall give two proofs of the case $0 < p < 1$ here and as we mentioned earlier, the first proof can be viewed as an analogue to Bennett's proof ([3, Theorem 1]) of Theorem 1.6 and the second proof is a generalization of the proof of Theorem 62 in [15]. An inspection of the proofs shows that they also work for the cases $p \geq 1$ and $p < 0$ as well (taken into account the reversed inequality of (2.3)). The first proof given below can also be viewed as a translation of the proof of Theorem 1.2 given in the previous section via duality. From now on in this section, we assume $0 < p < 1$.

The first proof:

Our discussion in Section 2 suggests that if we take the approach there, then we should take an auxiliary sequence (W_k) so that asymptotically, a similar expression would lead to something like the last expression of (2.2). One can see in what follows that our selection of the auxiliary sequence in the proof is then guided by this. By the reversed Hölder's inequality, we have

$$\begin{aligned}
&\Big(\sum_{k=n}^{\infty}\Big(\frac{(k+1)^r-k^r}{r}\Big)a_k\Big)^p \\
\geq\ & \Big(\sum_{k=n}^{\infty}\big(k^{r-1/p}-(k+1)^{r-1/p}\big)\Big)^{p-1}\cdot\Big(\sum_{k=n}^{\infty}\Big(\frac{(k+1)^r-k^r}{r}\Big)^p\big(k^{r-1/p}-(k+1)^{r-1/p}\big)^{1-p}a_k^p\Big) \\
=\ & n^{(1-rp)(1-p)/p}\sum_{k=n}^{\infty}\Big(\frac{(k+1)^r-k^r}{r}\Big)^p\big(k^{r-1/p}-(k+1)^{r-1/p}\big)^{1-p}a_k^p.
\end{aligned} \tag{3.1}$$

We then proceed as in Section 2 to see that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big(\frac{1}{n^r}\sum_{k=n}^{\infty}\Big(\frac{(k+1)^r-k^r}{r}\Big)a_k\Big)^p \\
\geq\ &\sum_{k=1}^{\infty}a_k^p\Big(\frac{(k+1)^r-k^r}{r}\Big)^p\Big(k^{r-1/p}-(k+1)^{r-1/p}\Big)^{1-p}\sum_{n=1}^{k}n^{1/p-(1+r)}.
\end{aligned}$$

It therefore suffices to show that

$$\Big(\frac{(k+1)^r-k^r}{r}\Big)^p\big(k^{r-1/p}-(k+1)^{r-1/p}\big)^{1-p}\sum_{n=1}^{k}n^{1/p-(1+r)} \geq \Big(\frac{p}{1-rp}\Big)^p. \tag{3.2}$$

We now apply inequality (2.3) to see that in order for inequality (3.2) to hold, it suffices to show that for $n \geq 1$ (note that for $(2+r)p \leq 1, 1/p-(1+r) \geq 1$),

$$\Big(\frac{(n+1)^r-n^r}{r}\Big)^p\Big(\frac{n^{r-1/p}-(n+1)^{r-1/p}}{(1-rp)/p}\Big)^{1-p} \geq \frac{n^{1+r-1/p}-(n+1)^{1+r-1/p}}{1/p-1-r}.$$

We can recast the above inequality as

$$\Big(\int_n^{n+1}x^{r-1}dx\Big)^p\Big(\int_n^{n+1}x^{r-1/p-1}dx\Big)^{1-p} \geq \int_n^{n+1}x^{r-1/p}dx. \tag{3.3}$$

Hölder's inequality now implies the above inequality and this completes the first proof.

The second proof:

Similar to (2.1), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big(\frac{1}{n^r}\sum_{k=n}^{\infty}\Big(\frac{(k+1)^r-k^r}{r}\Big)a_k\Big)^p \\
\geq\ &\sum_{k=1}^{\infty}a_k^p\Big(\frac{(k+1)^r-k^r}{r}\Big)^pW_k\sum_{n=1}^{k}\frac{1}{n^{rp}}\Big(\sum_{i=n}^{\infty}W_i^{1/(1-p)}\Big)^{p-1}.
\end{aligned}$$

We now choose W_k to be

$$W_k = \Big(\frac{(k+1)^r-k^r}{r}\Big)^{-p}\Big(\sum_{i=1}^{k}i^{\gamma}\Big)^{-1}, \quad \gamma = \frac{1-rp}{p}-1.$$

Using (2.3) and (3.3), we have

$$\begin{aligned}
W_k &\leq (1+\gamma)\Big(\int_k^{k+1}x^{r-1}dx\Big)^{-p}\Big(\int_k^{k+1}x^{-\gamma-1}dx\Big) \\
&\leq (1+\gamma)\Big(\int_k^{k+1}x^{-\gamma-2}dx\Big)^{1-p}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\Big(\frac{(k+1)^r-k^r}{r}\Big)^pW_k\sum_{n=1}^{k}\frac{1}{n^{rp}}\Big(\sum_{i=n}^{\infty}W_i^{1/(1-p)}\Big)^{p-1} \\
\geq\ &(1+\gamma)^{-1}\Big(\sum_{i=1}^{k}i^{\gamma}\Big)^{-1}\sum_{n=1}^{k}\frac{1}{n^{rp}}\Big(\sum_{i=n}^{\infty}\int_i^{i+1}x^{-\gamma-2}dx\Big)^{p-1} \\
=\ &(1+\gamma)^{-p} = \Big(\frac{p}{1-rp}\Big)^p.
\end{aligned}$$

This now completes the second proof of Theorem 1.1.

4. Proofs of Theorem 1.4 and Theorem 1.5

We first give the proof of Theorem 1.4 and we need a lemma:

Lemma 4.1. *Let $0 < p < 1, 0 < r \leq p$, $\beta \geq 1 + 2r/(1-p)$. The function*
$$f_{p,r,\beta}(x) = x^{-1}\Big((1+x)^{\beta - r/(1-p)} - (1+x)^{-r/(1-p)}\Big)$$
is an increasing function of $0 \leq x \leq 1$.

Proof. We have $f'_{p,r,\beta}(x) = x^{-2}(1+x)^{-r/(1-p)-1}h_{p,r,\beta}(x)$, where
$$h_{p,r,\beta}(x) = 1 + x - (1+x)^{\beta+1} + \frac{x}{1-p}\Big((\beta(1-p)-r)(1+x)^{\beta} + r\Big).$$
We also have
$$\frac{1-p}{1-p+r}h'_{p,r,\beta}(x) = \beta\Big(\frac{\beta(1-p)-r}{1-p+r}\Big)x(1+x)^{\beta-1} - \Big((1+x)^{\beta} - 1\Big) \geq 0,$$
where the last inequality above follows from the mean value theorem and our assumption on β. As $h_{p,r,\beta}(0) = 0$, it follows that $h_{p,r,\beta}(x) \geq 0$ for $0 \leq x \leq 1$. We then deduce from this that $f_{p,r,\beta}(x)$ is an increasing function of $0 \leq x \leq 1$ and this completes the proof. □

We now return to the proof of Theorem 1.4 and by a change of variables, $a_n \to n^{(r-p)/p}a_n$, we can recast inequality (1.3) as
$$\sum_{n=1}^{\infty}\frac{1}{n^r}\Big(\sum_{k=n}^{\infty}k^{(r-p)/p}a_k\Big)^p \geq c_{p,r}\sum_{n=1}^{\infty}a_n^p. \tag{4.1}$$
We follow the process in the first proof of Theorem 1.1 in Section 3, but this time, instead of using $k^{r-1/p} - (k+1)^{r-1/p}$ in (3.1), we use $k^{-\beta} - (k+1)^{-\beta}$, where $\beta > 0$ is a constant to be chosen later. The same process then leads to inequality (4.1) with the constant $c_{p,r}$ given by
$$\min_{k\geq 1}\Big(\big(k^{-\beta} - (k+1)^{-\beta}\big)^{1-p}k^{r-p}\sum_{n=1}^{k}n^{\beta(1-p)-r}\Big). \tag{4.2}$$
We note the following inequality ([15, Lemma 1, p. 18]), which asserts for $0 \leq r \leq 1$, we have
$$\sum_{i=1}^{n}i^r \geq \frac{n(n+1)^r}{1+r}. \tag{4.3}$$

We now assume $r/(1-p) \leq \beta \leq (1+r)/(1-p)$ so that $0 \leq \beta(1-p) - r \leq 1$ and we can use the bound (4.3) in (4.2) to see that
$$c_{p,r} \geq \min_{k\geq 1}\frac{f^{1-p}_{p,r,\beta}(1/k)}{1+\beta(1-p)-r},$$
where $f_{p,r,\beta}(x)$ is defined as in Lemma 4.1. We now take $\beta = 1 + 2r/(1-p)$ and it is easy to verify that the so chosen β satisfies $r/(1-p) \leq \beta \leq (1+r)/(1-p)$. It then follows from Lemma 4.1 that $\min_{k\geq 1} f^{1-p}(1/k) = \lim_{x\to 0^+} f^{1-p}(x) = \beta^{1-p} = (1+2r/(1-p))^{1-p}$. This now leads to the constant $c_{p,r}$ given in the statement of Theorem 1.4 and this completes the proof.

We now give the proof of Theorem 1.5. Again we follow the process in the first proof of Theorem 1.1 in Section 3 and similar to our proof of Theorem 1.4 above, instead of using $k^{r-1/p} - (k+1)^{r-1/p}$ in (3.1), we use $k^{-\beta} - (k+1)^{-\beta}$, where $\beta > 0$ is a constant to be determined. The same process then leads to inequality (1.2) with $c_{1/2} = \min_{k\geq 1}s_k$, where
$$s_k = \big(k^{-\beta} - (k+1)^{-\beta}\big)^{1/2}\sum_{n=1}^{k}n^{(\beta-1)/2}.$$

We want to choose β properly to maximize $c_{1/2}$. On considering s_1 and $\lim_{k\to+\infty} s_k$, we see that

$$c_{1/2} \geq \min\Big((1-2^{-\beta})^{1/2}, \frac{2\beta^{1/2}}{1+\beta}\Big).$$

Note that when $\beta=2$, $(1-2^{-2})^{1/2}=\sqrt{3}/2$ and when $\beta=3$, $2*3^{1/2}/(1+3)=\sqrt{3}/2$. As $(1-2^{-\beta})^{1/2}$ is an increasing function of β while $2\beta^{1/2}/(1+\beta)$ is a decreasing function of $\beta \geq 1$, our calculations above show that it suffices to consider $2 \leq \beta \leq 3$. On setting $(1-2^{-\beta})^{1/2} = 2\beta^{1/2}/(1+\beta)$, we find that the optimal β is approximately 2.4739 and the value of $(1-2^{-\beta})^{1/2}$ or $2\beta^{1/2}/(1+\beta)$ at this number is approximately 0.9055. This suggests that in order to maximize the value of $c_{1/2}$, we need to take β to be around 2.47. We now take $\beta = 2.4$ instead and use the bound (4.3) to see that

$$\big(k^{-\beta}-(k+1)^{-\beta}\big)^{1/2}\sum_{n=1}^{k} n^{(\beta-1)/2} \geq \frac{2}{1+\beta}u_{\beta}^{1/2}(1/k),$$

where

$$u_{\beta}(x) = x^{-1}\Big((1+x)^{\beta-1}-(1+x)^{-1}\Big).$$

We have $u'_{\beta}(x) = x^{-2}(1+x)^{-2}v_{\beta}(x)$, where

$$v_{\beta}(x) = 1+x-(1+x)^{1+\beta}+x(1+(\beta-1)(1+x)^{\beta}).$$

It's easy to check that $v'_{\beta}(0)=0$ and that

$$v''_{\beta}(x) = \beta(1+x)^{\beta-2}(\beta-3+(\beta^2-\beta-2)x).$$

It's also easy to see that the last factor of the right-hand expression above is <0 when $\beta=2.4$ and $0 \leq x \leq 1/3$. It follows that $v'_{\beta}(x) \leq 0$ when $\beta = 2.4$ and $0 \leq x \leq 1/3$. As $v_{\beta}(0)=0$, we deduce that $v_{\beta}(x) \leq 0$ when $\beta=2.4$ and $0 \leq x \leq 1/3$. This means that when $\beta = 2.4$, $u_{\beta}(x)$ is a decreasing function for $0 \leq x \leq 1/3$.

Our discussions above combined with direct calculations now imply that

$$c_{1/2} \geq \min\Big(\frac{2}{1+2.4}u_{2.4}^{1/2}(1/11) \approx 0.9001, \min_{1\leq k\leq 10} s_k\Big) \geq 0.9.$$

This completes the proof of Theorem 1.5.

5. Another proof of Theorem 1.6

By Hölder's inequality, we have

$$\Big(\sum_{k=1}^{n}\big(k^{\alpha}-(k-1)^{\alpha}\big)a_k\Big)^p \leq \Big(\sum_{k=1}^{n} W_k^{1/(1-p)}\Big)^{p-1}\Big(\sum_{k=1}^{n} W_k\big(k^{\alpha}-(k-1)^{\alpha}\big)^p a_k^p\Big),$$

where (W_k) is a sequence to be determined later. It follows that

$$\begin{aligned}&\sum_{n=1}^{\infty}\Big(\frac{1}{n^{\alpha}}\sum_{k=1}^{n}\big(k^{\alpha}-(k-1)^{\alpha}\big)a_k\Big)^p \\ \leq\ & \sum_{k=1}^{\infty} a_k^p\big(k^{\alpha}-(k-1)^{\alpha}\big)^p W_k \sum_{n=k}^{\infty}\frac{1}{n^{\alpha p}}\Big(\sum_{i=1}^{n} W_i^{1/(1-p)}\Big)^{p-1}.\end{aligned}$$

We now choose W_k to be

$$W_k = \big(k^{\alpha}-(k-1)^{\alpha}\big)^{-p}\Big(\sum_{i=k}^{\infty} i^{\gamma}\Big)^{-1}, \quad \gamma = -\frac{\alpha p-1}{p}-1.$$

We now note the following inequality ([3, (30)]) for $k \geq 1, \gamma > 1$:

$$(k^\gamma - (k-1)^\gamma)\sum_{n=k}^\infty \frac{1}{n^\gamma} \leq \frac{\gamma}{\gamma-1}. \tag{5.1}$$

The above estimation implies that

$$W_k \geq \alpha^{-p}\Big(\frac{p}{\alpha p-1}\Big)^{-1}\Big(\frac{k^\alpha-(k-1)^\alpha}{\alpha}\Big)^{-p}\int_{k-1}^k x^{\alpha-1/p}dx.$$

It follows that

$$\begin{aligned} W_k &\geq \alpha^{-p}\Big(\frac{p}{\alpha p-1}\Big)^{-1}\Big(\int_{k-1}^k x^{\alpha-1}dx\Big)^{-p}\int_{k-1}^k x^{\alpha-1/p}dx \\ &\geq \alpha^{-p}\Big(\frac{p}{\alpha p-1}\Big)^{-1}\Big(\int_{k-1}^k x^{\alpha-1/p-1}dx\Big)^{1-p}, \end{aligned}$$

where the last inequality above follows from Hölder's inequality. We then have

$$\Big(\sum_{i=1}^n W_i^{1/(1-p)}\Big)^{p-1} \leq \alpha^p\Big(\frac{p}{\alpha p-1}\Big)\Big(\sum_{i=1}^n\int_{i-1}^i x^{\alpha-1/p-1}dx\Big)^{p-1} = \alpha^p\Big(\frac{p}{\alpha p-1}\Big)^p n^{(\alpha-1/p)(p-1)}.$$

We then deduce that

$$\begin{aligned} &\big(k^\alpha-(k-1)^\alpha\big)^p W_k\sum_{n=k}^\infty\frac{1}{n^{\alpha p}}\Big(\sum_{i=1}^n W_i^{1/(1-p)}\Big)^{p-1} \\ \leq\ &\alpha^p\Big(\frac{p}{\alpha p-1}\Big)^p\Big(\sum_{n=k}^\infty n^{-1-\alpha+1/p}\Big)^{-1}\sum_{n=k}^\infty\frac{n^{(\alpha-1/p)(p-1)}}{n^{\alpha p}} = \Big(\frac{\alpha p}{\alpha p-1}\Big)^p. \end{aligned}$$

This now completes the proof of Theorem 1.6.

6. Weighted Remainder Form of Hardy-type Inequalities

In this section we study the weighted remainder form of Hardy-type inequalities in general. Let (λ_n) be a positive sequence satisfying $\sum_{n=1}^\infty \lambda_n < +\infty$. We set in this section $\Lambda_n = \sum_{k=n}^\infty \lambda_k$ and consider inequality (1.9). As we mentioned earlier, our goal is to find conditions on the λ_n's so that inequality (1.9) (or its reverse) can hold under these conditions. Our approaches in this section follow closely the approaches used in [7], [8] and [9]. We first let N be a large integer and for $1 \leq n \leq N$, we set $S_n = \sum_{k=n}^N \lambda_k a_k$ and

$$A_n = \frac{\sum_{k=n}^N \lambda_k a_k}{\Lambda_n}. \tag{6.1}$$

It follows from [6, (2.6)] that we have for $0 < p < 1, 1/p + 1/q = 1$, $1 \leq k \leq N$,

$$\mu_k S_k^{1/p} - (\mu_k^q - \eta_k^q)^{1/q} S_{k+1}^{1/p} \leq \eta_k \lambda_k^{1/p} a_k^{1/p}, \tag{6.2}$$

where $\mu_k^q > \eta_k^q \geq 0$ and the above inequality reverses when $p > 1$. Here we define $S_{N+1} = 0$. Due to similarities, we shall suppose $0 < p < 1$ here and summing the above inequality from $k = 1$ to N leads to

$$\mu_1 S_1^{1/p} + \sum_{k=2}^N\Big(\mu_k - (\mu_{k-1}^q - \eta_{k-1}^q)^{1/q}\Big)S_k^{1/p} \leq \sum_{n=1}^N \eta_n\lambda_n^{1/p}a_n^{1/p}.$$

We now set $\eta_i = \lambda_i^{-1/p}$ together with a change of variables $\mu_i \to \mu_i\eta_i$ to recast the above inequality as

$$\frac{\mu_1 S_1^{1/p}}{\lambda_1^{1/p}} + \sum_{k=2}^N \Big(\frac{\mu_k}{\lambda_k^{1/p}} - \frac{(\mu_{k-1}^q - 1)^{1/q}}{\lambda_{k-1}^{1/p}}\Big) S_k^{1/p} \leq \sum_{n=1}^N a_n^{1/p}.$$

We further set $\mu_i^q - 1 = \nu_i$ and make a further change of variables: $p \to 1/p$ to recast the above inequality as

$$\frac{(1+\nu_1)^{1-p} S_1^p}{\lambda_1^p} + \sum_{k=2}^N \Big(\frac{(1+\nu_k)^{1-p}}{\lambda_k^p} - \frac{\nu_{k-1}^{1-p}}{\lambda_{k-1}^p}\Big) S_k^p \leq \sum_{n=1}^N a_n^p.$$

We now set $\nu_i = \sum_{n=i+1}^\infty w_i/w_i$, where w_n's are positive parameters, to recast the above inequality as

$$\frac{w_1^{p-1}}{\lambda_1^p}\Big(\sum_{i=1}^\infty w_i\Big)^{1-p} \Lambda_1^p A_1^p + \sum_{n=2}^N \Big(\sum_{k=n}^\infty w_k\Big)^{-(p-1)} \Big(\frac{w_n^{p-1}}{\lambda_n^p} - \frac{w_{n-1}^{p-1}}{\lambda_{n-1}^p}\Big) \Lambda_n^p A_n^p \leq \sum_{n=1}^N a_n^p.$$

By a change of variables $w_n \to \lambda_n w_n^{1/(p-1)}$, we can recast the above inequality as

$$\frac{w_1}{\lambda_1}\Big(\sum_{i=1}^\infty \frac{\lambda_i w_i^{1/(p-1)}}{\Lambda_1}\Big)^{1-p} \Lambda_1 A_1^p + \sum_{n=2}^N \Big(\frac{\sum_{k=n}^\infty \lambda_k w_k^{1/(p-1)}}{\Lambda_n}\Big)^{-(p-1)} \Big(\frac{w_n}{\lambda_n} - \frac{w_{n-1}}{\lambda_{n-1}}\Big) \Lambda_n A_n^p \leq \sum_{n=1}^N a_n^p.$$

With another change of variables, $w_n/w_{n-1} \to b_n$ with $w_0 = 1$, we can further recast the above inequality as

$$\begin{aligned} &\frac{b_1}{\lambda_1}\Big(\frac{\sum_{k=1}^\infty \lambda_k \prod_{i=1}^k b_i^{1/(p-1)}}{\Lambda_n}\Big)^{-(p-1)} \Lambda_1 A_1^p \\ &+ \sum_{n=2}^N \Big(\frac{\sum_{k=n}^\infty \lambda_k \prod_{i=n}^k b_i^{1/(p-1)}}{\Lambda_n}\Big)^{-(p-1)} \Big(\frac{b_n}{\lambda_n} - \frac{1}{\lambda_{n-1}}\Big) \Lambda_n A_n^p \leq \sum_{n=1}^N a_n^p. \end{aligned} \tag{6.3}$$

We now choose the b_n's to satisfy:

$$\sum_{k=n}^\infty \lambda_k \prod_{i=n}^k b_i^{1/(p-1)} = \frac{p}{p-L}\Lambda_n.$$

From this we solve the b_n's to get

$$b_n = \Big(1 - \frac{L}{p}\frac{\lambda_n}{\Lambda_n}\Big)^{1-p}.$$

Upon requiring $\Lambda_n(b_n/\lambda_n - 1/\lambda_{n-1}) \geq 1 - L/p$ (with $1/\lambda_0 = 0$) and letting $N \to +\infty$, we deduce easily from (6.3) the $p > 1$ (and $0 < p < 1$) cases of the following

Theorem 6.1. *Let $p \neq 0$ be fixed and $a_n > 0$. Let L be a number satisfying $L < p$ when $p > 0$ and $L > p$ when $p < 0$. Suppose that $\lim_{n\to\infty} \Lambda_{n+1}(\sum_{k=n+1}^\infty \lambda_k a_k/\Lambda_{n+1})^p/\lambda_n = 0$ when $p < 0$. When $p > 1$ or $p < 0$, if (with $\Lambda_0/\lambda_0 = 1$) for $n \geq 1$,*

$$\frac{\Lambda_{n-1}}{\lambda_{n-1}} \leq \frac{\Lambda_n}{\lambda_n}\Big(1 - \frac{L\lambda_n}{p\Lambda_n}\Big)^{1-p} + \frac{L}{p}, \tag{6.4}$$

then inequality (1.9) *holds when $p > 1$ or $p < 0$. If the reversed inequality above holds when $0 < p < 1$, then the reversed inequality of* (1.9) *also holds.*

The case $p < 0$ of Theorem 6.1 follows from the same arguments above staring from inequality (6.2), as it still holds for $p < 0$, except this time we substitute S_n by $\sum_{k=n}^\infty \lambda_k a_k$ and A_n by $\sum_{k=n}^\infty \lambda_k a_k/\Lambda_n$. In this case, we may assume $\sum_{k=n}^\infty \lambda_k a_k/\Lambda_n < +\infty$, for otherwise, inequality (1.9) holds automatically.

On taking Taylor expansion of the right-hand side expression of (6.4), we deduce easily from Theorem 6.1 the following

Corollary 6.1. *Let $p \neq 0$ be fixed and $a_n > 0$. Let L be a number satisfying $L < p$ when $p > 0$ and $L > p$ when $p < 0$. Suppose that $\lim_{n\to\infty} \Lambda_{n+1}(\sum_{k=n+1}^{\infty}\lambda_k a_k/\Lambda_{n+1})^p/\lambda_n = 0$ when $p < 0$. When $p > 1$ or $p < 0$, if (with $\Lambda_0/\lambda_0 = 1$) for $n \geq 1$,*

$$L \geq \frac{\Lambda_{n-1}}{\lambda_{n-1}} - \frac{\Lambda_n}{\lambda_n}, \tag{6.5}$$

then inequality (1.9) *holds when $p > 1$ or $p < 0$. If the reversed inequality above holds when $0 < p < 1$, then the reversed inequality of* (1.9) *also holds.*

We now give an improvement of the above result:

Theorem 6.2. *Let $p \neq 0$ be fixed and $a_n > 0$. Let L be a number satisfying $L < p$ when $p > 0$ and $L > p$ when $p < 0$. Suppose that $\lim_{n\to\infty} \Lambda_{n+1}(\sum_{k=n+1}^{\infty}\lambda_k a_k/\Lambda_{n+1})^p/\lambda_n = 0$ when $p < 0$. When $p \geq 1$ or $p < 0$, if (with $\Lambda_0/\lambda_0 = 1$) for $n \geq 1$, inequality* (6.5) *holds, then for $p \geq 1$,*

$$\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{\lambda_k a_k}{\Lambda_n}\Big)^p \leq \frac{p}{p-L}\sum_{n=1}^{\infty}a_n\Big(\sum_{k=n}^{\infty}\frac{\lambda_k a_k}{\Lambda_n}\Big)^{p-1}.$$

The above inequality reverses when $p < 0$. When $0 < p \leq 1$, the reversed inequality above also holds if the reversed inequality (6.5) *holds for all $n \geq 1$.*

Proof. We consider the cases $p \geq 1$ and $0 < p \leq 1$ first. Due to similarities, we assume $p \geq 1$ here. We let N be a large integer and start with the inequality $x^p - px + p - 1 \geq 0$, valid for $x > 0, p \geq 1$ or $p < 0$ with the reversed inequality being valid for $x > 0, 0 < p \leq 1$. On setting $x = A_{n+1}/A_n$, $1 \leq n \leq N$ with $A_{N+1} = 0$, where $A_n, 1 \leq n \leq N$ is defined as in (6.1), we obtain

$$A_{n+1}^p + (p-1)A_n^p \geq pA_{n+1}A_n^{p-1}. \tag{6.6}$$

Note that

$$A_{n+1} = \frac{\Lambda_n A_n}{\Lambda_{n+1}} - \frac{\lambda_n a_n}{\Lambda_{n+1}}.$$

Substituting this expression of A_{n+1} on the right-hand side of (6.6), we obtain after some simplifications that

$$\Big(\frac{\Lambda_n}{\lambda_n} + p - 1\Big)A_n^p - \Big(\frac{\Lambda_n}{\lambda_n} - 1\Big)A_{n+1}^p \leq pa_nA_n^{p-1}.$$

Summing the above inequality from $n = 1$ to N, we obtain

$$\sum_{n=1}^{N}\Big(\frac{\Lambda_n}{\lambda_n} - \frac{\Lambda_{n-1}}{\lambda_{n-1}} + p\Big)A_n^p \leq p\sum_{n=1}^{N}a_nA_n^{p-1}. \tag{6.7}$$

The assertion of the theorem for the cases $p \geq 1$ now follows easily from the case $N \to +\infty$ of the above inequality and inequality (6.5).

The case $p < 0$ of the assertion of the theorem follows from the same arguments above, except this time we substitute A_n by $\sum_{k=n}^{\infty}\lambda_k a_k/\Lambda_n$. In this case, we may assume $\sum_{k=n}^{\infty}\lambda_k a_k/\Lambda_n < +\infty$, for otherwise, the assertion of the theorem holds automatically. $\square$

When $\sum_{n=1}^{\infty}a_n^p < +\infty$ and that $\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\lambda_k a_k/\Lambda_n\Big)^p < +\infty$, then by Hölder's inequality, we have for $p > 1$,

$$\sum_{n=1}^{\infty}a_n\Big(\sum_{k=n}^{\infty}\frac{\lambda_k a_k}{\Lambda_n}\Big)^{p-1} \leq \Big(\sum_{n=1}^{\infty}a_n^p\Big)^{1/p}\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{\lambda_k a_k}{\Lambda_n}\Big)^p\Big)^{1/q}.$$

with the above inequality reversed when $0 \neq p < 1$ and from which one easily deduces the assertion of Corollary 6.1. Note that when $p > 0$, one can also deduce the assertion of Corollary 6.1 without assuming $\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\lambda_k a_k/\Lambda_n\Big)^p < +\infty$. Since one can start with (6.7), repeat the argument above and then let $N \to +\infty$.

We now study the following so called weighted remainder form of Carleman-type inequality, corresponding to the limiting case $p \to +\infty$ of inequality (1.9) (after a change of variables $a_n^p \to a_n$):

$$\sum_{n=1}^{\infty}\Big(\prod_{k=n}^{\infty} a_k^{\lambda_k/\Lambda_n}\Big) \leq E\sum_{n=1}^{\infty} a_n. \tag{6.8}$$

This is first studied by Pečarić and Stolarsky in [17, Sect 3]. Our starting point is the following result of Pečarić and Stolarsky [17, (3.5)], which is an outgrowth of Redheffer's approach in [18]:

$$\sum_{n=1}^{N}\Lambda_n(b_n-1)G_n + G_1\Lambda_1 - \Lambda_{N+1}G_{N+1} \leq \sum_{n=1}^{N}\lambda_n a_n b_n^{\Lambda_n/\lambda_n}, \tag{6.9}$$

where N is a large integer, $\mathbf{b}$ is any positive sequence and

$$G_n = \prod_{k=n}^{\infty} a_k^{\lambda_k/\Lambda_n}.$$

We now make a change of variables $\lambda_n a_n b_n^{\Lambda_n/\lambda_n} \to a_n$ to recast inequality (6.9) as

$$\begin{aligned}&\sum_{n=1}^{N}\Lambda_n(b_n-1)\Big(\prod_{k=n}^{\infty}\lambda_k^{-\lambda_k/\Lambda_n}\Big)\Big(\prod_{k=n}^{\infty}b_k^{-\Lambda_k/\Lambda_n}\Big)G_n + G_1\Lambda_1\Big(\prod_{k=1}^{\infty}\lambda_k^{-\lambda_k/\Lambda_n}\Big)\Big(\prod_{k=1}^{\infty}b_k^{-\Lambda_k/\Lambda_n}\Big)\\ &-G_{N+1}\Lambda_{N+1}\Big(\prod_{k=N+1}^{\infty}\lambda_k^{-\lambda_k/\Lambda_{N+1}}\Big)\Big(\prod_{k=N+1}^{\infty}b_k^{-\Lambda_k/\Lambda_{N+1}}\Big) \leq \sum_{n=1}^{N} a_n.\end{aligned}$$

Now, a further change of variables $b_n \to \lambda_{n-1}b_n/\lambda_n$ with $\lambda_0 > 0$ an arbitrary number allows us to recast the above inequality as

$$\frac{\Lambda_1 b_1 G_1}{\lambda_1}\prod_{k=1}^{\infty}b_k^{-\Lambda_k/\Lambda_n} + \sum_{n=2}^{N}\Lambda_n\Big(\frac{b_n}{\lambda_n}-\frac{1}{\lambda_{n-1}}\Big)G_n\prod_{k=n}^{\infty}b_k^{-\Lambda_k/\Lambda_n} - \frac{\Lambda_{N+1}G_{N+1}}{\lambda_N}\prod_{k=N+1}^{\infty}b_k^{-\Lambda_k/\Lambda_{N+1}} \leq \sum_{n=1}^{N} a_n. \tag{6.10}$$

If we now choose the values of b_n's so that $\prod_{k=n}^{\infty} b_k^{-\Lambda_k/\Lambda_n} = e^{-M}$, we then solve the b_n's to get $b_n = e^{M\lambda_n/\Lambda_n}$ and upon substituting these values for b_n's we obtain via (6.10):

$$\frac{\Lambda_1}{\lambda_1}e^{M\lambda_1/\Lambda_1}G_1 + \sum_{n=2}^{N}\Lambda_n\Big(\frac{e^{M\lambda_n/\Lambda_n}}{\lambda_n}-\frac{1}{\lambda_{n-1}}\Big)G_n - \frac{\Lambda_{N+1}}{\lambda_N}G_{N+1} \leq e^M\sum_{n=1}^{N} a_n. \tag{6.11}$$

We immediately deduce from (6.11) the following

Theorem 6.3. *Suppose that* $\lim_{n\to\infty}\Lambda_{n+1}G_{n+1}/\lambda_n = 0$ *and that (with* $\Lambda_0/\lambda_0 = 1$*)*

$$M = \sup_{n\geq 1}\frac{\Lambda_n}{\lambda_n}\log\Big(\frac{\Lambda_{n-1}/\lambda_{n-1}}{\Lambda_n/\lambda_n}\Big) < +\infty,$$

then inequality (6.8) *holds with* $E = e^M$.

We note that

$$\log\Big(\frac{\Lambda_{n-1}/\lambda_{n-1}}{\Lambda_n/\lambda_n}\Big) = \log\Big(1+\frac{\Lambda_{n-1}/\lambda_{n-1}-\Lambda_n/\lambda_n}{\Lambda_n/\lambda_n}\Big) \leq \frac{\Lambda_{n-1}/\lambda_{n-1}-\Lambda_n/\lambda_n}{\Lambda_n/\lambda_n}.$$

It follows from this and Theorem 6.3 that we have the following

Corollary 6.2. *Suppose that* $\lim_{n\to\infty}\Lambda_{n+1}G_{n+1}/\lambda_n = 0$ *and that (with* $\Lambda_0/\lambda_0 = 1$*)*

$$M = \sup_{n\geq 1}\Big(\frac{\Lambda_{n-1}}{\lambda_{n-1}}-\frac{\Lambda_n}{\lambda_n}\Big) < +\infty,$$

then inequality (6.8) *holds with* $E = e^M$.

We now consider another choice for the b_n's in (6.10) by setting $b_n = e^{(\Lambda_{n-1}/\lambda_{n-1}-\Lambda_n/\lambda_n)/(\Lambda_n/\lambda_n)}$ with $\Lambda_0/\lambda_0 = 1$ and it follows from this and (6.10) that

$$\sum_{n=1}^{N}\Big(\frac{\Lambda_n e^{(\Lambda_{n-1}/\lambda_{n-1}-\Lambda_n/\lambda_n)/(\Lambda_n/\lambda_n)}}{\lambda_n}-\frac{\Lambda_{n-1}}{\lambda_{n-1}}+1\Big)G_n e^{-\sum_{k=n}^{\infty}\frac{\lambda_k}{\Lambda_n}(\frac{\Lambda_{k-1}}{\lambda_{k-1}}-\frac{\Lambda_k}{\lambda_k})}$$
$$-\frac{\Lambda_{N+1}G_{N+1}}{\lambda_N}e^{-\sum_{k=N+1}^{\infty}\frac{\lambda_k}{\Lambda_{N+1}}(\frac{\Lambda_{k-1}}{\lambda_{k-1}}-\frac{\Lambda_k}{\lambda_k})} \leq \sum_{n=1}^{N} a_n,$$

from which we deduce the following

Corollary 6.3. *Suppose that* $\lim_{n\to\infty} \Lambda_{n+1}G_{n+1}/\lambda_n = 0$ *and that (with* $\Lambda_0/\lambda_0 = 1$*)*

$$M = \sup_{n\geq 1}\sum_{k=n}^{\infty}\frac{\lambda_k}{\Lambda_n}(\frac{\Lambda_{k-1}}{\lambda_{k-1}}-\frac{\Lambda_k}{\lambda_k}) < +\infty,$$

then inequality (6.8) *holds with* $E = e^M$.

Note that the above corollary also implies Corollary 6.2. We now consider some applications of our results above. When $\lambda_n = n^{\alpha} - (n+1)^{\alpha}$, $-1 \leq \alpha < 0$, by Lemma 1 (and property (iv) of $f_{\alpha}(x)$ defined there) of [3], we have for $n \geq 1$,

$$\frac{n^{\alpha}}{n^{\alpha}-(n+1)^{\alpha}}-\frac{(n+1)^{\alpha}}{(n+1)^{\alpha}-(n+2)^{\alpha}} \leq \frac{1}{\alpha}. \tag{6.12}$$

One can show also easily that $1-\alpha \geq 2^{-\alpha}$ for $-1 \leq \alpha < 0$ and that for fixed $-1 \leq \alpha < 0$. Moreover, we have $(n+1)^{\alpha}G_{n+1}/(n^{\alpha}-(n+1)^{\alpha}) \leq (n+1)G_{n+1}/(-\alpha)$, so that it follows from Corollary 6.2 that we have the following

Corollary 6.4. *Let* $-1 \leq \alpha < 0$ *and assume that* $\lim_{n\to\infty} nG_n = 0$, *then*

$$\sum_{n=1}^{\infty}\Big(\prod_{k=n}^{\infty}a_k^{(k^{\alpha}-(k+1)^{\alpha})/n^{\alpha}}\Big) \leq e^{1/\alpha}\sum_{n=1}^{\infty}a_n. \tag{6.13}$$

The constant is best possible.

By taking $a_n = n^{-1-\epsilon}$ with $\epsilon \to 0^+$, one shows that the constant in (6.13) is indeed best possible. We note that inequality (6.12) is reversed when $\alpha \leq -1$ and we also have $1-\alpha \leq 2^{-\alpha}$ when $\alpha \leq -1$ and it follows from Corollary 6.1 that this gives another proof of the case $r \leq -1, 0 < p < 1$ of inequality (1.4). We point out that similar to the treatment in inequality (1.7), one can show the case $r \leq -1, 0 < p < 1$ of inequality (1.4) also follows from inequality (2.3) and the validity of inequality (1.8) for $\alpha \geq 2, p < 0$.

When $\lambda_n = n^{\alpha}$, $\alpha < -1$, it follows from (5.1) with $\gamma = -\alpha$, we have for $n \geq 1$,

$$\frac{\sum_{i=n}^{\infty} i^{\alpha}}{n^{\alpha}}-\frac{\sum_{i=n+1}^{\infty} i^{\alpha}}{(n+1)^{\alpha}} \geq \frac{1}{\alpha+1}.$$

Note that the case $k = 1$ of (5.1) with $\gamma = -\alpha$ also implies the reversed inequality (6.5) when $n = 1$ and it follows from Corollary 6.1 that we have the following

Corollary 6.5. *Let* $\alpha < -1$ *and* $a_n > 0$, *then for* $0 < p < 1$,

$$\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}\frac{k^{\alpha}a_k}{\sum_{i=n}^{\infty} i^{\alpha}}\Big)^p \geq \Big(\frac{(1+\alpha)p}{(1+\alpha)p-1}\Big)^p\sum_{n=1}^{\infty}a_n^p. \tag{6.14}$$

The constant is best possible.

By taking $a_n = n^{-1/p-\epsilon}$ with $\epsilon \to 0^+$, one shows that the constant in (6.14) is indeed best possible. We point out here that similar to the treatment for the $p>1$ case of Theorem 1.1 given in Section 1, we can recast inequality (6.14) via the duality principle as (by a change of variable $\alpha \to -\alpha$),

$$\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}\frac{\alpha}{(\alpha-1)(k^{\alpha}-(k-1)^{\alpha})\sum_{i=k}^{\infty}i^{-\alpha}}\frac{(k^{\alpha}-(k-1)^{\alpha})a_k}{n^{\alpha}}\Big)^p \leq \Big(\frac{\alpha p}{\alpha p-1}\Big)^p\sum_{n=1}^{\infty}a_n^p.$$

Here we have $\alpha>1$ and $p<0$. It is then easy to see that the above inequality (hence inequality (6.14)) also follows from inequality (5.1) and the $p<0$ case of inequality (1.6).

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